

Algebraic Reasoning in the Elementary Classroom: Results of a Professional
Development Program for Teachers

Susan Jo Russell

TERC

Deborah Schifter

Education Development Center

Virginia Bastable

Math Leadership Program, Mt. Holyoke College

Megan Franke

University of California, Los Angeles

Author Note

Susan Jo Russell, TERC, Cambridge, Massachusetts; Deborah Schifter, Education Development Center, Waltham, Massachusetts; Virginia Bastable, Math Leadership Program, Mount Holyoke College, South Hadley, Massachusetts; Megan Franke, Department of Education, University of California, Los Angeles.

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TERC, 2067 Massachusetts Avenue, Cambridge, MA 02144.

E-mail: Susan_Jo_Russell@terc.edu

Abstract

This paper describes a professional development experience designed to help teachers support their students to make, represent, and justify generalizations about the behavior of the operations. We report on the pre- and post-assessments of 36 teachers who participated in an online course and implemented what they were learning in their classrooms across a full school year, and of the 600 students in their classrooms versus a comparison group. Participating teachers improved significantly in articulating general claims about the operations, representing mathematical ideas, and using mathematical language and notation. Their students, particularly those in grades 3-5, provided significantly more relational explanations on post-assessment items than comparison students. We consider what characteristics of the professional development may account for these results.

Keywords: Generalization, proof, mathematical argument, representation, relational explanation, professional development, online learning

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Third-grade teacher, Alice Kaye, presents the following poster to her class.

$7 + 5 = 12$	$7 + 5 = 12$
$7 + 6 = \underline{\quad}$	$8 + 5 = \underline{\quad}$
$9 + 4 = 13$	$9 + 4 = 13$
$9 + 5 = \underline{\quad}$	$10 + 4 = \underline{\quad}$
<p>What do you notice?</p> <p>What's happening here?</p>	

Ms. Kaye acknowledges to the students that the numbers are not challenging, but that the purpose of the discussion is to consider what is going on in the pairs of problems and to state ideas clearly and convincingly. The class fills in the blanks, and then talks about what they notice. Students mention that one number changes, one stays the same, and the last number in the equation changes, too. The discussion goes on for a few minutes, until Evan says, “Since $9 + 4$ is 13, $9 + 5$ *has* to be 1 more than 13.”

After further discussion, Pamela says, “I was just wondering. How did Evan come up with the idea he had? Because these are not just everyday ideas that you come up with

every day.”

Evan responds, “I’m not really sure. I just know it. It kind of seems obvious to me, so I didn’t think to think about it before.”

This discussion among third graders launched an exploration of generalizations about the behavior of the operations. They first worked together to state a conjecture about the effect on the sum of adding 1 to an addend. They represented the conjecture, and then justified why it must be true for all whole numbers. Later, they considered what happens when an addend is increased by any amount, and then what happens when analogous changes are made with other operations.

Such work—which brings *operations* to the fore and leaves *numbers* in the background—is an aspect of early algebra. The equation $a + (b + 1) = (a + b) + 1$, which captures the students’ conjecture in symbolic form, is a statement about the behavior of addition regardless of the values of a and b .

This paper describes a professional development experience designed to help teachers support their students to make, represent, and justify generalizations about the behavior of the operations. Teachers participated in an online course and implemented what they were learning in their classrooms across a full school year. We report on student and teacher learning results and consider what characteristics of the professional development may account for them.

Historical Context

The work presented here is related to two trends in mathematics education over the last two decades: 1) the introduction of algebra in the elementary grades, and 2) greater emphasis on pedagogical practices that support students’ mathematical reasoning.

In the early 1990s, a great deal of publicity was given to the idea that algebra was a gateway to educational and professional opportunities. One response was to consider how to better prepare students for algebra through their early mathematics studies. Some research teams introduced elementary-grade students to variations of problems that typically appear in Algebra I. Others investigated young students' ability to engage with the basic ideas of functions. Still others worked on generalized arithmetic. Cai and Knuth (2011) and Kaput, Carraher, and Blanton (2007) edited volumes that describe many of these efforts.

The second trend, a shift in pedagogical practices to put greater emphasis on students' mathematical reasoning, was captured in *Principles and Standards for School Mathematics*, published by the National Council of Teachers of Mathematics (NCTM) in 2000, and in other documents that followed (NCTM, 2000, 2003; National Research Council, 2001). These initiated a nationwide discussion of how teaching practices might emphasize students' understanding and flexible use of basic mathematical concepts, their ability to solve novel problems, and their engagement with mathematical ideas. More recently, NCTM's *Principles to Actions* (NCTM, 2014) highlights eight key mathematics teaching practices that support the development of mathematical understanding for all students. These include two that are particularly relevant to the work presented here—*use and connect mathematical representations* and *facilitate meaningful mathematical discourse*.

The work of the project presented in this paper—preparing teachers to support their students to make, represent, and justify generalizations about the behavior of the operations—builds on both of these movements. The authors have focused on

generalizations that grow out of elementary students' core work in computation (Schifter, 1999; Russell, Schifter, & Bastable, 2006; Schifter, Bastable, Russell, Riddle & Seyferth, 2008). We have found that when students are encouraged to verbalize their mathematical questions and observations about the operations, they frequently become curious about regularities they notice. Discussion of these regularities can engage a wide range of learners who tap into the ideas from different perspectives. Further, we believe that this focus on the operation as an object of study helps students unpack the structure of each operation—how it behaves and how it differs from other operations. For example, students have traditionally been brought into the operation of multiplication as repeated addition, and many elementary teachers think of multiplication in this way. While the connection with the accumulative nature of addition may be a natural entry point into multiplication, students also need to come to understand that multiplication behaves in ways quite different from addition.

In our projects, students work with such generalizations as the properties of the operations, the relationship between addition and subtraction, and the relationship between multiplication and division, as well as other generalizations that may be derived from these properties and relationships—e.g., given a multiplication expression, if one factor is doubled and the other is halved, the product is unchanged. Familiarity with these generalizations supports students' fluency in calculation. These ideas, which will later be communicated through algebraic notation, also underlie algebraic manipulation used to create equivalent expressions and solve equations.

A Classroom Example

The classroom scene described above launches a sequence of lessons that illustrates what it means for elementary students to make, represent, and justify general claims about operations. At the end of that first discussion, Ms. Kaye gives the class directions to do one of two things: 1) to write down a statement if they have a way of putting an idea about the regularity they are noticing into words or 2) to come up with other pairs of equations that show the same regularity.

Ms. Kaye goes through the students' work and creates two posters, one with more examples, the other with ways students tried to articulate the idea. Of the attempts to make a general claim, one of the clearest is, "If an equation has an answer that is one more than the other equation, one of the numbers has to go up."

On the second day, the class works together to create a statement that will communicate their idea to someone who hasn't been present for the conversation. They settle on: "In addition, if you increase one of the addends by 1, then the sum will also increase by 1."

In a later session, Ms. Kaye presents the challenge to use a story context, picture, diagram, or manipulatives to convince somebody else that this conjecture is true for all whole numbers. The video clip below shows a short segment of the discussion when students come back together, after working in pairs to devise convincing arguments.

[PLACE LINK FOR VIDEO CLIP HERE (2:52)]

After they work on addition, the class turns to some lessons on multiplication.

They explore what happens when 1 is added to a factor and consider how addition and multiplication behave differently.

There are three major phases in the lesson sequence for each operation.

1. The teacher presents problems to draw students' attention to a generalization about the behavior of the operation. The point of the discussion is not to solve the problems, but to notice relationships across problems.
2. The class works together to articulate the idea in words. Once students describe what they notice in the examples and begin to explain the relationships involved, they formulate a conjecture. Expressing the generalization is challenging for many students. They require focused time to develop a clear statement.
3. The class uses visual representations that embody the behavior of the operation to represent their conjecture. Although their demonstrations use particular numbers of cubes, students explain how their cube stacks can represent any number and show why their claim must work, no matter what numbers they start with.

Young students do not have access to the tools of formal mathematical proof, but they can still learn how explanations and justifications of mathematical claims can be based on more than examples or assertions. Representations—drawings, diagrams, physical models, and story contexts—are important tools for constructing mathematical arguments about the operations. In order to be effective, these representations must show the actions and structure of the operation, that is, they indicate how different quantities are related and demonstrate what happens when one state is transformed into another. In order to develop an argument about their conjecture, students in Ms. Kaye's class used

drawings and physical models that embodied the structure of the operation of addition, showing the relationship between addends and sum. These models became the basis for their explanations about why adding 1 to an addend must result in the sum increasing by 1. We refer to such student justifications as *representation-based argument*.

The sequence of activities described here engages learners in different aspects of structure and regularity of the operations. For students like Evan, the child who said the idea was obvious, the challenge is to communicate the conjecture with precision and to prove it. Students who have not developed the habit of noticing regularities get a chance to think about them explicitly. Some students benefit from representing the operations in ways that strengthen their grasp of what the operations do and how they behave. All of the students have the opportunity to deepen their understanding of what an operation is and how each operation is different from the others. This work emphasizes that an operation is not just an instruction to perform a computation, but is a mathematical object with particular characteristics.

Professional Development Goals and Structure

For the project, *Foundations of Algebra*, we designed a professional development program to bring teachers into the ideas illustrated by Ms. Kaye's lessons. The goals were to help teachers:

- understand and look for generalizations implicit in their students' work in number and operations,
- bring students' attention to such generalizations,
- help students articulate general conjectures,
- have students create visual representations of their conjectures as a step

toward proving them,

- create mathematical explanations and arguments based on those representations, and
- attend to the range of learners in the class as they engage in these activities.

To these ends, teachers investigated mathematical ideas and representations for themselves, learned to attend more closely to their students' mathematical thinking, and worked to create a classroom community of inquiry.

The professional development was offered as an online seminar for teachers in school-based teams. There were two online components—weekly online postings of assignments and responses to these by participants, and six synchronous webinars across the year. So that participants could attend to issues of teaching and learning throughout the school year, the seminar ran from September to May in three intensive six-week blocks. The one- to two-month breaks between blocks allowed teachers to solidify mathematical ideas and pedagogical practices before encountering new challenges.

Participants engaged in three main kinds of activity: discussing chapters from the course text, doing mathematics activities designed for adult learners, and writing *student thinking assignments* analyzing their efforts to engage their students with course ideas (Russell, Schifter, & Bastable, 2011b).

The book, *Connecting Arithmetic to Algebra* (Russell, Schifter, & Bastable, 2011a), formed the basis of the professional development. Each chapter includes classroom examples written by teachers focused on an aspect of integrating generalizations about the operations into classroom instruction. Reading the text was an

opportunity for participants to dig into mathematics content, student thinking, classroom dynamics, and teacher moves. After reading each chapter, participants wrote about their reactions to specific passages and connections to their own classrooms.

The mathematics activities involved examining the behavior of the operations, articulating conjectures, and developing mathematical arguments. The mathematics assignments served two purposes: participants learned mathematics content needed to work on these ideas with their own students and they experienced learning as a process of refining their own ideas through engagement with the ideas of others. Here is one assignment:

Examine the truth of this statement:

“Adding 1 to one of the addends in an addition expression results in the total being increased by 1.” For example, since $16 + 7 = 23$, then $16 + 8 = 24$.

Begin by examining some more examples of this statement. Then use at least two different representations to illustrate your examples. Can you use your representations to show why the statement is true in general, that is, for any pair of positive whole numbers?

Do the same for this statement, “Adding 1 to one of the numbers in a subtraction expression results in adding 1 to the answer.”

The activities of noticing, articulating, and representing generalizations were new to participants. In their work together, they compared their use of language and their representations, particularly considering what it might mean to construct a generalization for any pair of whole numbers.

Student thinking assignments provided the link between the ideas of the seminar and participants' classrooms by requiring participants to engage their own students in activities of generalizing and justifying. Participants recorded mathematics discussions and interviews with students, selected portions to transcribe, and wrote a narrative about each episode, including their thoughts about the students' thinking and their teaching moves. For example, the first assignment was to work on questions such as these with their students:

Is this number sentence true? $19 + 6 = 20 + 5$

How do you know they are equal? 3×16 and 6×8

As you plan, think about how you will help your students learn to have mathematical discussions. Record the class discussion. When you listen to the recording, note student responses that particularly intrigue, surprise, or please you. Choose a passage to describe in detail in writing, including actual student dialogue. Write about two or three of the students' responses.

Throughout the course, participants posted to an online forum their responses to the chapters in the text, their work on the mathematics problems, and their student thinking assignments. They were put in groups of three or four in order to respond to

each other's postings.. In addition, facilitators wrote detailed comments in response to each teacher's student thinking assignments. The different kinds of activity—readings, mathematics assignments, and student thinking assignments—all addressed the same set of ideas from different perspectives, offering multiple entry points and different kinds of challenges.

Teacher Learning

A study of the teacher participants' learning was designed to capture teachers' own knowledge about generalizations about the operations, their understanding of how elementary students think about generalizing and justifying, and their knowledge of pedagogical moves related to algebraic thinking.

Participants

Teachers in the study were participants in the second iteration of the Connecting Arithmetic to Algebra course (2010-2011).¹ In the Participant Group, we included the 36 classroom teachers (not specialists or coaches) for whom we had complete student data (see Student Learning section) and both pre and post teacher assessment data. A Comparison Group consisted of 16 teachers who had not taken the course but worked in the same school systems as course participants.

Teacher Problem Set

Teachers in both groups completed a problem set at the beginning and end of the school year. The Teacher Problem Set included items that asked teachers to generate potential claims that might arise in elementary school classrooms, create representations to support a given claim, make sense of a claim made by a student, analyze students'

¹ Five online sections were largely taught by facilitators who had apprenticed during the first iteration. One of the co-PIs continued to supervise the new group of facilitators and co-facilitated two sections.

justifications for a claim, produce questions to ask a student in relation to a particular algebraic idea, and use algebraic notation. With the exception of the one item that focused on teachers' knowledge of symbolic notation to express mathematical generalizations, all problems were posed in a context of student thinking or classroom instruction. The items and scoring rubric were piloted and revised during the first year the online course was given (2009-2010).

Results

Teachers in the Participant Group changed significantly from pretest to posttest in three areas: the breadth of claims they could generate and articulate, their representation of mathematical ideas, and their use of mathematical language and notation.

1. Generating and Articulating Claims

Figure 1. shows one of the two tasks that focused on generating and articulating claims.

In a second grade classroom, students noticed that they could switch the order of the numbers in an addition problem and still get the same sum. For example, when solving $4 + 18$, students would change the order to $18 + 4$ to make it easier to solve. This is one example of a general claim that might come up in the course of students' work in computation. Can you think of three other correct general claims that might come up in grades 1-6 classrooms?

Figure 1. Teacher assessment item for generating and articulating claims

Figure 2 shows an example of one participant's pre-course response.

Can you think of three other correct general claims that might come up in grades 1-6 classrooms?

1. $4 \times 18 = 18 \times 4$
2. $0 + 1 = 1$
 $0 + 4 = 4$
 $0 + 9 = 9$
3. $0 \times 3 = 0$
 $0 \times 6 = 0$
 $0 \times 9 = 0$

Figure 2. One teacher's pretest response to the sample item on generating and articulating claims

This example is a comparatively good pre-course response. Underlying each response, one can see what generalization the teacher might have been thinking about, although the teacher doesn't actually articulate it. The teacher here gets credit for generating these three claims but is scored as providing only examples with no other articulation. Most pre-course responses were not as complete or clear as this one, and fewer than 20% of the teachers could generate three correct claims. Figure 3 shows the post-course response from the same participant.

1. commutative property of multiplication - you can change the order of the factors in a multiplication expression & get the same sum
 $a \times b = b \times a$

2. commutative property does NOT work for subtraction or division
 - the order of the subtrahend & minuend matter
 - the order of the divisor & dividend matter

3. if you subtract an amount from one addend in an addition expression & add the same amount to the remaining addend, the sum remains the same (and vice versa)

1. Commutative property of multiplication—you can change the order of the factors in a multiplication expression and get the same sum² $a \times b = b \times a$.

2. Commutative property does not work for subtraction or division—the order of the subtrahend and minuend matter; the order of the divisor and dividend matter.

3. If you subtract an amount from one addend in an addition expression and add the same amount to the remaining addend, the sum remains the same (and vice versa).

Figure 3. One teacher's posttest response to the sample item on generating and articulating claims

As illustrated by the response above, teachers improved in their ability to articulate a claim completely with sufficient detail and precise language. The posttest response specifies the operation in each of the generalizations; each statement is clear

² We note the incorrect use of the word “sum” here, but the notation and the rest of the wording make the intention of the participant's response clear.

about what changes, how it changes, and what the result is; and the third claim uses an if-then structure. Participants improved significantly in their ability to generate three correct claims [Stuart-Maxwell test of marginal homogeneity $X^2_{35;.05}=21.93$]; at pretest 12% of the teachers could do so compared to 88% at posttest.

2. Representing Mathematical Ideas

Participants were asked to generate representations related to a general claim (see Figure 4).

Some students generated this conjecture:

If you double one number in a multiplication expression and cut the other number in half, you get the same answer, for example, $12 \times 5 = 6 \times 10$.

1. Provide another example of this conjecture.
2. Show two representations that would be useful in supporting this claim (for all whole numbers).

Figure 4. Teacher assessment item on generating representation for a claim

Figure 5 shows one example—again a relatively good example—of a teacher’s pretest response.

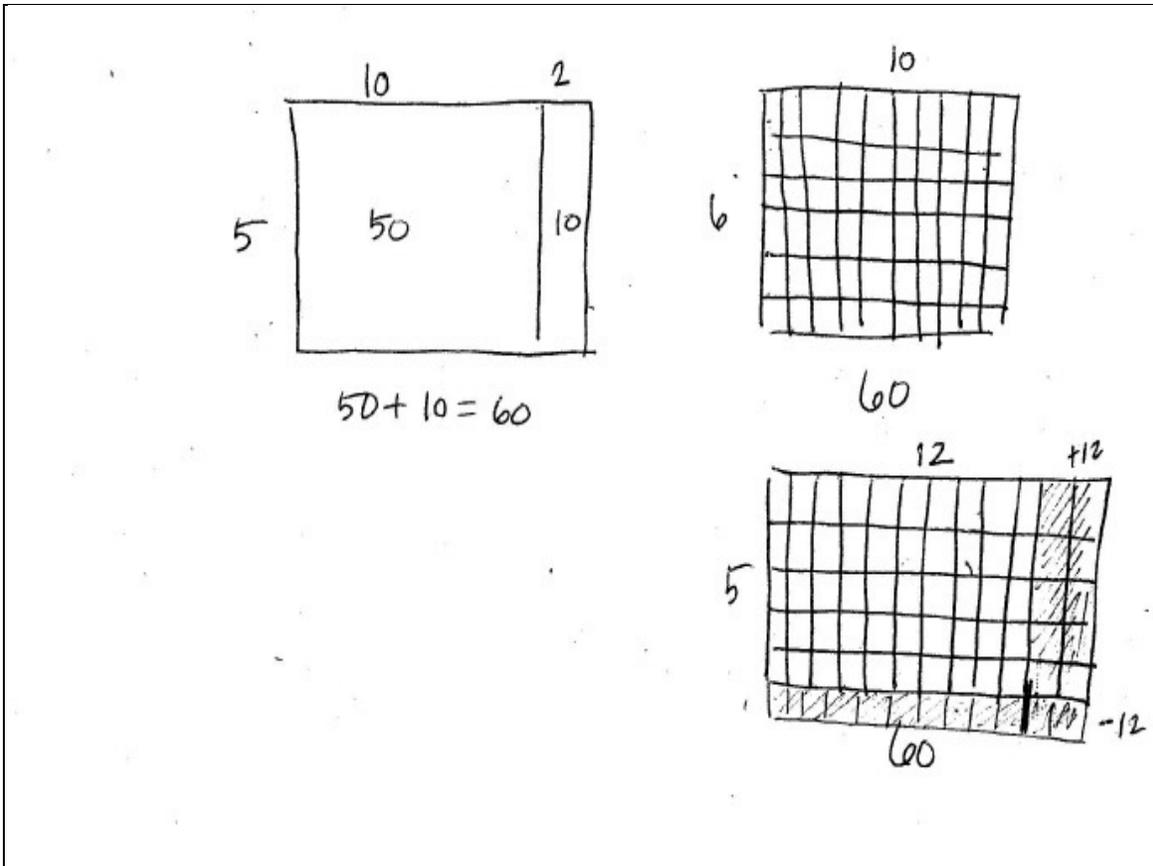


Figure 5. Teacher pretest response to generating representation item, $12 \times 5 = 6 \times 10$

The picture shows area representations of the two expressions separately in the top row. The bottom picture superimposes the two rectangles and shows how 6×10 can be transformed into 5×12 by first adding two columns of 6 (creating a 6×12 rectangle), then subtracting one row of 12 (to create the 5×12 rectangle). Since the same amount (12) is first added, then subtracted, the resulting rectangle has an area equivalent to the original rectangle.

This pretest response demonstrates knowledge about representing multiplication. However, although the representation shows that the two areas are the same for this particular case, it does not explain the generalization that doubling one factor and halving the other maintains the product. Rather than halving 12 to become 6 and doubling 5 to become 10, in this picture 10 is increased by 2 to 12 and 6 is decreased by 1 to 5.

On the posttest, the same participant showed three representations of the claim (see Figure 6).

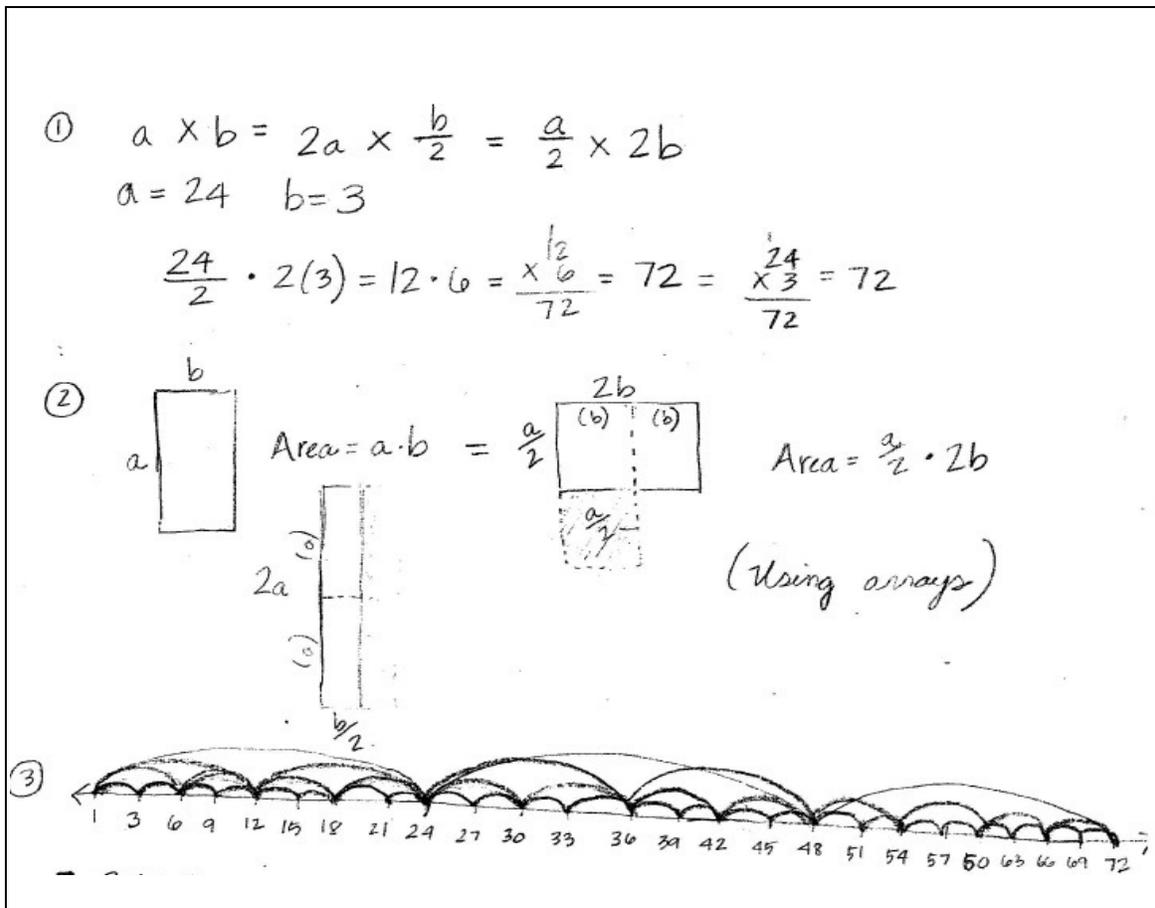


Figure 6. Teacher posttest response to generating representation item, $12 \times 5 = 6 \times 10$

While the second representation has some aspects in common with the area model used on the pretest, this diagram shows how the rectangle $a \times b$ is transformed into the rectangle $a/2 \times 2b$ (halving side a and doubling side b) by cutting the original rectangle to create two $a/2 \times b$ pieces and moving one of those pieces, thus maintaining the same area. It also shows how the original rectangle becomes $2a \times b/2$ by aligning two rectangles of dimensions $a \times b/2$. The third representation shows how halving the length of the equal jumps on the number line, thereby doubling the number of jumps, leads to the same result.

At pretest only 11% of the teachers provided two useful representations; 42% provided at least one. At posttest 36% of the teachers provided two appropriate representations, and 63% provided at least one. From pretest to posttest, the Participant Group improved significantly in their ability to represent operations and claims about the operations [Stuart-Maxwell test of Marginal Homogeneity $X^2_{34;.05} = 8.93$].

Improvement in teachers' own use of representation was complemented by significant changes in how they thought about using representation in their instruction. For example, a problem that asked teachers to respond to a student who is thinking about the different identity properties of addition and multiplication did not require teachers to reference any representations, but the use of representation would certainly be appropriate and helpful. At pretest, less than half (42%) of the teachers mentioned the use of representation, and, of those, only 19% described how to draw on them extensively and in detail. At posttest 72% of the teachers mentioned the use of representation, and 47% of those teachers used them extensively.

Although teaching the use of algebraic notation was not a specific goal of the course, participants encountered examples of its use in student work, and a chapter of the text focused on some of the affordances and challenges in the use of such notation. On the pretest, 5% of the teachers used variables to represent a claim, while at posttest, 25% did so (as in Figures 3 and 6). Further, participants improved significantly on two specific items asking them to use algebraic notation. At pretest 42% of teachers could correctly represent the claim, "If you add an amount to one addend and subtract the same amount from another addend, the sum does not change," while at posttest, 69% could do

so. Clearly, many participants were already able to use algebraic notation before the course began, and with exposure to correct use of notation, more learned to do so.

3. *Using Mathematical Language*

In assessing teachers' use of mathematical language, we were looking for the extent to which they used careful and mathematically appropriate language to make arguments or detail claims. All items asking for articulation or explanation were examined together for each teacher according to the following rubric:

- 0: little if any use of mathematical language and/or unclear and incorrect use of language
- 1: occasional use of mathematical language
- 2: consistent use of mathematical language

Improvement was significant in this area (Stuart-Maxwell test of Marginal Homogeneity $X^2_{34,05} = 27.7$) as shown in Table 1.

Table 1. Teacher Pretest and Posttest Language Scores

Language Score			
	0	1	2
Pre-test	69%	31%	0%
Post-test	5%	42%	53%

The 53% of teachers receiving a score of 2 for language at posttest used clear and purposeful natural and mathematical language throughout the assessment. They used mathematical terms (e.g., *addend*, *product*, *commutativity*, *factor*) correctly and efficiently to marshal their arguments. For example, one teacher consistently made statements like: “If you double one factor and halve the other factor in a multiplication problem the product remains the same. Here she draws on mathematical language to carefully articulate her claim.” The teachers not drawing on mathematical language in this way often used terms such as *flip flop*, *turn around*, and other less precise mathematical language, wrote little, or talked around an idea in a way that made it difficult to interpret their arguments. One teacher (receiving a 0 for language) regularly stated her ideas by using brief, unelaborated phrases such as, “order doesn’t matter” or “plus one, subtract one.”

Comparison Group Results

The Comparison Group did not improve significantly at any given task from pretest to posttest and there was no overall significant difference between the Comparison group and the Participant Group. However, the small number of Comparison participants indicated that this part of our analysis was underpowered. In looking at raw scores, there is evidence of contrasting responses between the two groups. For instance, on the item that asked teachers to create representations for a claim, 79% of the Comparison Group could not provide any at posttest, while 37% of the Participant teachers could not produce any representations at posttest. Similarly, none of the comparison teachers received a language score that indicated consistent use of mathematical language on either the pretest or posttest, and 71% of the teachers in the comparison group remained at the

lowest level of language use at posttest. Yet, in examining the total score for teachers there was no difference between the two groups ($p > .05$).

While we need to be cautious about the claims we make here, the data suggest that the teachers who participated in professional development did not improve on the items simply due to the passing of time or test retake since the Comparison Group made no such improvement.

Student Learning

Students were assessed to see whether and how participation of their teachers in the professional development was evidenced in students' algebraic learning.

Participants

Student assessments were administered by the teachers in the Participant Group and the Comparison Group. Pre and post data were collected for students in the 36 Participant classrooms and 16 Comparison classrooms.

Student Problem Set

The Early Algebra Student Assessment (EASA) was designed to capture students' algebraic thinking in ways consistent with the goals of the professional development. It was piloted in a number of contexts and then fully piloted during the first iteration of the course.

Items asked students to (a) compare expressions and explain why they think two are equal, using words and pictures; (b) explain and evaluate a claim made by another student and then use that claim to solve a problem; (c) evaluate equations as true or false related to the properties of arithmetic and use of notation; and (d) write a word problem that matched a given numerical expression (see Table 2 for examples).

Two versions of the assessment were designed: one for grades K-2, the other, grades 3-5. The items overlapped across these two versions except that the grade K-2 students answered three fewer compare-expression items and wrote a word problem for a subtraction expression ($12 - 3$); grade 3-5 students wrote a problem for a multiplication expression (6×8).

Table 2. Examples of EASA Student Assessment Items

Level	Compare Expressions	Explain/Evaluate Claim (2 items)	Evaluate True/ False Equations (10 items)
Grade K-2	<p><i>Which are equal?</i> <i>Circle the ones that are equal.</i></p> <p>9 – 5 5 – 9 10 – 6</p>	<p>Ms. Garcia’s class was solving $35 + 12 = 25 + \underline{\quad}$. Ann said she knows it is 22 and she did not even need to add $35 + 12$. Ann said, “since 25 is 10 less than 35, I added 10 to the 12.”</p>	<p><i>Which are true?</i> <i>True or Not</i></p> <p>$3 + 5 = 5 + 3$ $a = a$ $15 = 3 \times 5$ $2 + 6 = 4 + 4$ $2 + 6 - 8 = 4 + 4$ $a + b = b - a$ $5 + 3 = 8 + 1 = 9$</p>
Grade 3-5	<p><i>Which are equal?</i> <i>Circle the ones that are equal.</i></p> <p>13×14 $(10 \times 10) + (3 \times 4)$ $(10 + 4) \times 13$</p>	<p>How did Ann know 22 goes in the blank without adding up 35 and 12?</p> <p>Use Ann’s method to solve this problem. Explain what you did.: $46 + 17 = 36 + \underline{\quad}$</p>	

Pre-Post EASA Results

We recognized several potential challenges to capturing any changes in student learning: a) the professional development was online, reaching teachers across the country, without face-to-face contact with the facilitators; b) the student data were

collected in the same year the professional development was occurring; and c) we were limited to using a paper and pencil measure.

Despite these challenges, pretest to posttest data captured changes in the algebraic thinking of Participant Group students. Table 3 shows the average scores of both the K-2 and 3-5 groups increased significantly (K-2 ($t_{127,.05}=10.34$) 3-5 ($t_{472,.05}=18.20$)).

Table 3. Average Scores on EASA Pre and Posttests for K-2 and 3-5 Grade Students in the Participant Group Classrooms

Grade	Pretest	Posttest
K-2 (n=128)		
(possible 24)	9.3	13.7*
3-5 (n=473)		
(possible 29)	16.4	20.5*

* $p < .05$

Since we would expect students to learn over the course of the year, we compared the item means of the Participant group second graders at *posttest* (after engaging in the intervention) to the third graders from Participant classrooms at *pretest* (with no intervention) where we would expect on average the students in these groups to look

alike. The percent correct was higher for the second grade posttest than the third grade pretest for many of the items (see Figure 7).

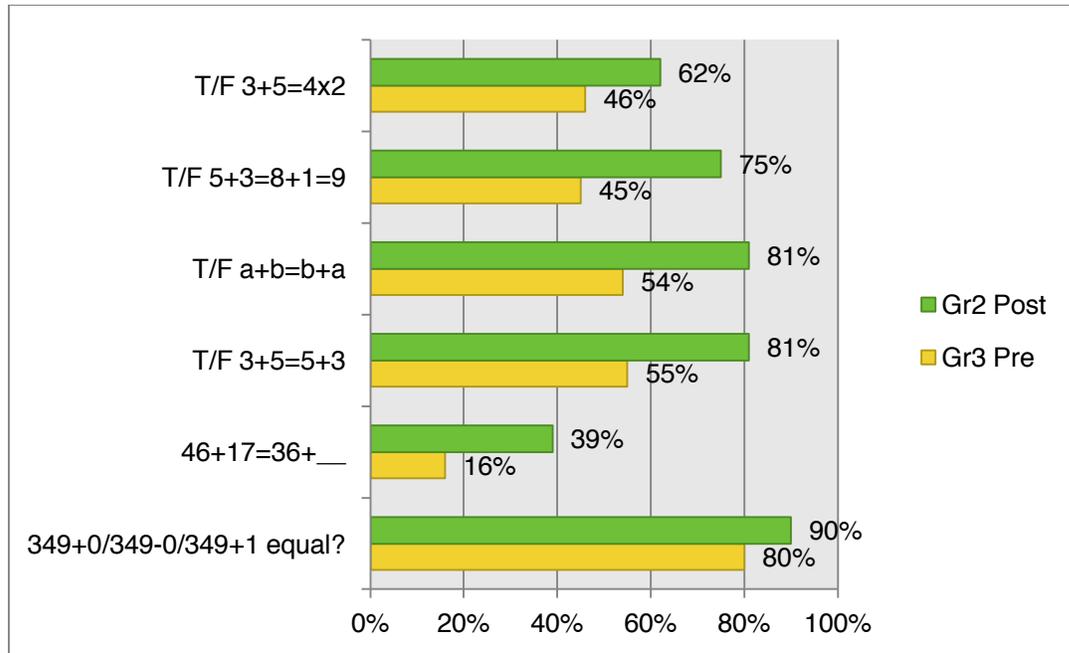


Figure 7. Percent Correct on selected EASA Post Gr 2 and Pretest Gr 3 items

We collected the same pre and posttest EASA data from the Comparison Group. The 3-5 grade students in the Participant Group outperformed the Comparison Group students ($t_{661; .05}=2.90$). However, the K-2 student scores did not differ by group ($p > .05$).

The most compelling data in relation to the goals of the professional development come from a comparison of student responses on the items on which students were asked to explain their thinking. On these items student responses were coded for the type of explanation they provided: a) no explanation; b) a computational explanation; or c) a relational explanation, that is, an explanation that uses a mathematical relationship. For instance, to explain why $9 - 5$ and $10 - 6$ are equal, a student could carry out both

computations, showing that each expression equals 4, or the student could give a relational explanation (Figure 8).

Which are equal? Circle the ones that are equal.

EXPLAIN using words and pictures how you know which are equal without trying to find the answer to each one.

5. $9 - 5$ $5 - 9$ $10 - 6$

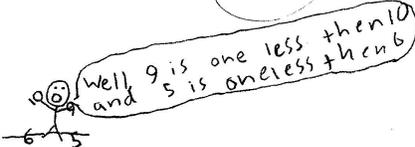
A handwritten student response. On the left, there is a stick figure with a circle for a head and a vertical line for a body. To the right of the stick figure is a speech bubble containing the text: "Well 9 is one less than 10 and 5 is one less than 6". Below the stick figure, the numbers "6" and "5" are written and connected by a horizontal line.

Figure 8. Student relational response to EASA item

On the items for which students were asked to explain their thinking (See Appendix A), students of teachers in the Participant Group, particularly those in third through fifth grades, provided significantly more relational explanations (see Table 4).

Table 4. Proportion of Students by Grade and Group Providing Relational Explanations at Posttest

Item	Comparison 3-5	Participant Gr 3-5	Comparison Gr 2	Participant Gr 2
9-5/10-6	0.05	0.26*	0.00	0.04
6×7/3×14	0.13	0.29*	NA	NA
27+9-9=27	0.36	0.68*	0.44	0.50
35+12=25+___	0.11	0.31*	0.02	0.18*
46+17=36+___	0.50	0.55	0.27	0.36

*(Grade 2 - $X^2_{1,05} = 5.9$) (Grade 3 -5 $X^2_{1,05} = 37.75; 57.50; 19.6; 28.7$)

The first two items in Table 4 required students to identify two of three expressions that were equivalent. The other three items were embedded in story contexts.

Conclusion

In the year-long online course, teachers learned about what it means to make generalizations in mathematics and how elementary students can engage in articulating, representing, and justifying general claims about the operations. The results of our assessments indicate that the activities of the professional development described in this article had an impact on teachers' understanding of this mathematical content and that what the teachers brought back to their classrooms had an impact on student learning.

What might account for these results?

The online format for professional development naturally imposed distance between facilitators and participants—they were not, after all, in the same room. Furthermore, there was no direct classroom observation or coaching. Still, teacher learning not only occurred, but what was learned was implemented in teachers' classrooms. The design and content of the seminar incorporated a number of features that have been highlighted in previous research as critical to teachers' learning:

- The seminar focused on specific mathematical content and on how students develop understanding of that content (Cohen & Hill, 1998; Hiebert et al., 1996; Kennedy, 1998).
- Seminar structures promoted “active learning” (Garet et al., 2001) in which teachers linked their learning to analysis of student work and classroom implementation.
- The seminar sustained work across a full school year, allowing teachers time to develop their own understanding and incorporate it into their practice. (Darling-Hammond et al., 2009; Garet, Porter, Desimone, Birman, & Yoon, 2001; Timperley, Wilson, Barrar, & Fung, 2007). The three 6-week parts of the course engaged participants intensely in study of mathematics content and student thinking. The two 6-week periods between the parts of the course provided time for participants to consolidate their ideas, implement them in the classroom, and learn from their interactions with their students.
- A variety of structures situated participants in a community of practice (Flint, Zisook, & Fisher, 2011). Depending on the assignment, they worked in different peer groupings: with groups of about 20 in the webinar, with their school team,

with their online response group, or with an online grade-level team.

Besides these factors, feedback from participants indicated that, despite the online context in which participants and facilitators never actually met, *participant-facilitator relationships were critical*. The correspondence between facilitators and participants around each of the nine student thinking assignments provided a direct and intimate connection. The assignments required participants to share and reflect on events from their own teaching. In order to write individual responses to each of these assignments, facilitators carefully considered each participant's learning trajectory and addressed specific details of each participant's writing. As the year went on, facilitators had time to determine which issues would be fruitful to pursue with each individual, offering encouragement and posing new questions or challenges as participants developed their thinking and practice. Upon receiving responses, participants had further opportunity to reflect and, because the interaction was sustained over time, could act on these reflections. Though attenuated through the online medium, these interactions established a relational stance of respect and caring (Flint et al., 2011; Shroyer, Yahnke, Bennett, & Dunn, 2007; Stein, Hubbard, & Mehan, 2002; Swars, Meyers, Mays, & Lack, 2009).

An instructional focus on the behavior of the operations is foundational to both arithmetic and algebra and provides students with a conceptual link between them. As illustrated in Ms Kaye's class, students are considering the operations as mathematical objects with their own properties and behaviors, not as only instructions to compute. As these students widen their understanding of the number system—from whole numbers to fractions and decimals, later to integers and beyond—a firm grasp of the properties and

behaviors of each operation, not dependent on the particular values being operated on, is critical.

In the process of studying the operations in this way, students engage in key mathematical practices (Cuoco, Goldenberg, & Mark, 1996; Kazemi & Hintz, 2014; National Council of Teachers of Mathematics, 2014; National Governors Association, 2010). They have opportunities to notice regularity in the behavior of an operation, articulate conjectures based on initial observations, investigate the structure of the operation that accounts for the behavior through creating representations of the mathematical relationships, develop arguments about why it holds, and critique each other's reasoning. Our results suggest that an instructional focus on representation-based argument supports students to learn how to develop and communicate mathematical explanations, which is increasingly seen as supportive of student learning (Webb et al., 2008, 2014).

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Appendix A. Relational Thinking Items

Items coded for Relational Thinking
1. Explain using words and pictures how you know which are equal without trying to find the answer to each one. [9-5 5-9 10-6]
2. Explain using words and pictures how you know which are equal without trying to find the answer to each one. [6x7 3x14 5x8]
3. Marissa says, “I like problems that go $27 + 9 - 9 = 27$ cause they are so easy. Because when you add a number and then take that number away you don't really have to do anything.” Do you agree with Marissa? Do you think it always works?
4. Ms. Garcia’s class was solving $35 + 12 = 25 + \underline{\quad}$. Ann said she knows it is 22 and she did not even need to add $35 + 12$. Ann said, “since 25 is 10 less than 35, I added 10 to the 12.” How did Ann know 22 goes in the blank without adding up 35 and 12?
5. Use Ann’s method to solve this problem. Explain what you did. $46 + 17 = 36 + \underline{\quad}$